

Gravitational Collapse in Higher Dimension

Ujjal Debnath* and Subenoy Chakraborty†

Department of Mathematics, Jadavpur University, Calcutta-32, India.

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Spherically symmetric inhomogeneous dust collapse has been studied in higher dimensional space-time and appearance of naked singularity has been analyzed both for non-marginal and marginally bound cases. It has been shown that naked singularity is possible for any arbitrary dimension in non-marginally bound case. For marginally bound case we have examined the radial null geodesics from the singularity and found that naked singularity is possible upto five dimension.

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I. INTRODUCTION

For the last two decades or so, gravitational collapse is an important and challenging issue in Einstein gravity, particularly after the formation of famous singularity theorems [1] and Cosmic Censorship Conjecture (CCC) [2]. Also it is interesting to know the final outcome of gravitational collapse [3] in the background of general relativity from the perspective of black hole physics as well as its astrophysical implications. The singularity theorems provide us only about the generic property of space times in classical general relativity but these theorems can not predict the detailed features of the singularities such as their visibility to an external observer as well as their strength. The CCC, on the other hand is incomplete [4,5] as it stands because there is no formal proof of it in one hand and on the other there are counter examples of it. However, the nature of the central shell focusing singularity depends on the choice of the initial data [6].

It has recently been pointed out by Joshi et al [7] that the physical feature which is responsible for the formation of naked singularity is nothing but the presence of shear. It is the shear developing in the gravitational collapse, which delays the formation of the apparent horizon so that the communication is possible from the very strong gravity region to observers situated outside.

The objective of this paper is to fully investigate the situation in the background of higher dimensional space-time with both non-marginally and marginally bound collapse [8-10]. As it is only to be expected, in one way or another, these works all deal with propagation of null geodesics in the space-time of a collapsing dust [12]. In this context we mention that Ghosh and Beesham [10] have also studied dust collapse for $(n+2)$ dimensional Tolman-Bondi space-time for marginally bound case ($f = 0$), considering the self-similar solutions. They have concluded that higher dimensions are favourable for black holes rather than naked singularities. Also recently, Ghosh and Banerjee [11] have considered non-marginal case ($f \neq 0$) for dust collapse in 5D Tolman-Bondi model and have shown that the degree of inhomogeneity of the collapsing matter is necessary to form a naked singularity.

We show in the present paper that in non-marginally bound collapse (i.e., for $f \neq 0$), the naked singularity may appear in any dimensional space-time, but for marginally bound collapse the naked singularity may appear only when the space-time has dimensions upto five.

II. NON-MARGINALLY BOUND CASE

The metric ansatz for n dimensional space time is

*Electronic address: ujjaldebnath@yahoo.com

†Electronic address: subenoyc@yahoo.co.in

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2 d\Omega_{n-2}^2 \quad (1)$$

with $d\Omega_{n-2}^2 = d\theta_1^2 + \sin^2\theta_1[d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \dots + \sin^2\theta_{n-3}d\theta_{n-2}^2)]$

We consider here a dust collapse with energy-momentum tensor:

$$T_{\mu\nu} = \rho(t, r)u_\mu u_\nu, \quad (2)$$

with u_μ as the n -velocity.

Now from the Einstein's field equations for the metric (1) with the above energy-momentum tensor, one can obtain (choosing $8\pi G = c = 1$)

$$\dot{R}^2 = \frac{F(r)}{R^{n-3}} + f(r), \quad (3)$$

and

$$\rho(t, r) = \frac{(n-2)F'(r)}{2R^{n-2}R'}, \quad (4)$$

where $F(r)$ is an arbitrary functions of r , arising from the integration with respect to the proper time t .

Suppose, the collapse develops from an initial surface $t = t_i$ and the above model is characterized by the initial density $\rho_i(r) = \rho(t_i, r)$ and $f(r)$, which describes the initial velocities of collapsing matter shells. We choose the scaling of the scale factor R such that

$$R(t_i, r) = r, \quad (5)$$

so that

$$\rho_i(r) = \rho(t_i, r) = \frac{n-2}{2}r^{2-n}F'(r) \quad (6)$$

Now the curve $t = t_s(r)$ defines the shell-focusing singularity and is characterized by

$$R(t_s(r), r) = 0 \quad (7)$$

Further, within the collapsing cloud, the trapped surfaces will be formed due to unbounded growth of the density and these trapped surfaces are characterized by the outgoing null geodesics. In fact, the apparent horizon, which is the boundary of the trapped surface has the equation $t = t_{ah}(r)$ and the scale factor at the apparent horizon satisfies

$$R(t_{ah}(r), r) = [F(r)]^{\frac{1}{n-3}} \quad (8)$$

To characterize the nature of the singularity we shall discuss the two possibilities namely, (i) $t_{ah} < t_s(0)$ and (ii) $t_{ah} > t_s(0)$. The first case may correspond to formation of black hole while the second one may lead to naked singularity. If the apparent horizon will form earlier than the instant of the formation singularity, then the event horizon can fully cover the strong gravity region and also the singularity. As a result, no light signal from the singularity can reach to any outside observer and the singularity is totally hidden within a black hole. On the other hand, in

the second case the trapped surfaces will form much later during the evolution of the collapse and it is possible to have a communication between the singularity and external observers.

Integrating once, we have from equation (3)

$$t - t_i = \frac{2}{(n-1)\sqrt{F}} \left[r^{\frac{n-1}{2}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{fr^{n-3}}{F}\right] - R^{\frac{n-1}{2}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{fR^{n-3}}{F}\right] \right] \quad (9)$$

where we have used the initial condition (5) and ${}_2F_1$ is the usual hypergeometric function with $a = \frac{1}{2} + \frac{1}{n-3}$. Using equations (7) and (8) separately in equation (9) we have

$$t_s(r) - t_i = \frac{2}{(n-1)\sqrt{F}} r^{\frac{n-1}{2}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{fr^{n-3}}{F}\right] \quad (10)$$

and

$$t_{ah}(r) - t_i = \frac{2r^{\frac{n-1}{2}}}{(n-1)\sqrt{F}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{fr^{n-3}}{F}\right] - \frac{2F^{\frac{1}{n-3}}}{n-1} {}_2F_1\left[\frac{1}{2}, a, a+1, -f\right] \quad (11)$$

We note that in case of homogeneous dust, the collapse is simultaneous (Openheimer-Snyder) but in the present inhomogeneous model the collapse is not simultaneous (in comoving co-ordinates) but rather the singularity is described by a curve with starting point $(t_0, 0)$, which is given by equation (10) as

$$t_0 = t_s(0) = t_i + \lim_{r \rightarrow 0} \frac{2r^{\frac{n-1}{2}}}{(n-1)\sqrt{F}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{fr^{n-3}}{F}\right] \quad (12)$$

Now in order to have a finite value of the above limit, we assume $F(r)$ and $f(r)$ [13,14] to be in the following polynomial form near the central singularity ($r = 0$)

$$F(r) = F_0 r^{n-1} + F_1 r^n + F_2 r^{n+1} + \dots \quad (13)$$

and

$$f(r) = f_0 r^2 + f_1 r^3 + f_2 r^4 + \dots \quad (14)$$

Then equation (5) suggests that initial density profile is also smooth at the centre and we have

$$\rho_i(r) = \rho_0 + \rho_1 r + \rho_2 r^2 + \dots \quad (15)$$

with $\rho_j = \frac{(n+j-1)(n-2)}{2} F_j$, $j = 0, 1, 2, \dots$

As the density gradient is negative and falls off rapidly to zero near the centre so we must have $\rho_1 = 0$ and $\rho_2 < 0$ and consequently $F_1 = 0$ and $F_2 < 0$. Thus the above limit (see eq.(12)) simplifies to

$$t_0 = t_i + \frac{2}{(n-1)\sqrt{F_0}} {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{f_0}{F_0}\right] \quad (16)$$

Therefore, equation (11) and (16) give (after using (13) and (14) in (11))

$$t_{ah} - t_0 = -\frac{2}{n-1} F_0^{\frac{1}{n-3}} r^{\frac{n-1}{2}} - \frac{f_1}{(3n-7)F_0^{3/2}} {}_2F_1\left[\frac{3}{2}, a+1, a+2, -\frac{f_0}{F_0}\right] r + \left[-\frac{F_2}{(n-1)F_0^{3/2}} \right]$$

$$\begin{aligned}
& {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{f_0}{F_0}\right] + \frac{1}{4(n-3)(3n-7)f_0F_0^{5/2}(f_0+F_0)} \left\{ (7-3n)f_0^2F_0^2 {}_2F_1\left[\frac{1}{2}, a, a+1, -\frac{f_0}{F_0}\right] \right. \\
& + \{-2f_0F_0((5-2n)f_1^2 + 2(n-3)f_2F_0) - 4(n-3)f_0^2F_0(f_2-F_2) + 4(n-3)f_0^3F_2 \\
& \left. + (3n-7)f_1^2F_0^2\} {}_2F_1\left[\frac{3}{2}, a+1, a+2, -\frac{f_0}{F_0}\right] \right\} r^2 + O(r^3) \quad (17)
\end{aligned}$$

From the above relation we note that near $r = 0$, $t_{ah} > or \leq 0$ according as

$$f_1 {}_2F_1\left[\frac{3}{2}, a+1, a+2, -\frac{f_0}{F_0}\right] < or \geq 0.$$

Though the hypergeometric function has arguments depending on n yet ${}_2F_1$ is always positive. So the above restriction simply implies $f_1 < or \geq 0$. Now according to Joshi et al, if a comoving observer (at fixed r) does not encounter any trapped surfaces until the time of singularity formation then it is possible to visualize the singularity otherwise the singularity is covered by trapped surfaces, leading to a black hole. Thus $f_1 < 0$ leads to naked singularity while $f_1 \geq 0$ is the restriction for black hole. However, the above conditions are not sufficient, particularly for non-smooth initial density profile. Therefore, we may have locally naked singularity or black hole depending on the sign of f_1 .

Further, to study the effect of shear on the formation of trapped surface we first evaluate the shear explicitly. For n -dimensional spherically symmetric dust metric the shear scalar is estimated by [7]

$$\sigma = \sqrt{\frac{n-2}{2(n-1)}} \left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} \right)$$

Using equation(3) we get in turn

$$\sigma = \sqrt{\frac{n-2}{8(n-1)}} \frac{[\{RF' - (n-1)R'F\} + R^{n-3}(Rf' - 2R'f)]}{R^{\frac{n-1}{2}} R' (F + fR^{n-3})^{1/2}} \quad (18)$$

Since at the initial hypersurface ($t = t_i$) we have chosen $R(t_i, r) = r$, so the initial shear σ_i is of the form

$$\sigma = \sqrt{\frac{n-2}{8(n-1)}} \frac{[\{rF' - (n-1)F\} + r^{n-3}(rf' - 2f)]}{r^{\frac{n-1}{2}} (F + fR^{n-3})^{1/2}}$$

Thus using the power series expansions (13) and (14) for $F(r)$ and $f(r)$ in the above expression for σ_i we have

$$\begin{aligned}
\sigma_i &= \sqrt{\frac{n-2}{8(n-1)}} \frac{f_1 r + \sum_{m=2}^{\infty} m(f_m + F_m)r^m}{\sqrt{(f_0 + F_0) + f_1 r + \sum_{m=2}^{\infty} m(f_m + F_m)r^m}} \\
&= \sqrt{\frac{n-2}{8(n-1)}} \frac{1}{\sqrt{f_0 + F_0}} \left[f_1 r + \left\{ 2(f_2 + F_2) - \frac{f_1^2}{2(f_0 + F_0)} \right\} r^2 + O(r^3) \right] \quad (19)
\end{aligned}$$

We note that the initial shear vanishes when $f_1 = 0$, $(F_2 + f_2) = (F_3 + f_3) = \dots = 0$ and hence even if the initial shear is zero the dust distribution may be inhomogeneous. Thus from equation (10) t_s a function of the comoving radial co-ordinate r , so that the shell focusing singularity appears at different r at different instants. Also the above expression for σ_i reveals that the existence of the naked singularity is not directly related to the non-vanishing of the shear as it does in the marginally bound case (see ref.[14]).

III. marginally bound case

In this case (i.e., $f = 0$) equation (3) can be integrated out easily to give

$$R^{\frac{n-1}{2}} = r^{\frac{n-1}{2}} - \frac{n-1}{2} \sqrt{F(r)}(t - t_i) \quad (20)$$

where we have used the initial condition (5).

Suppose the radius of the spherical shell R shrinks to zero at the time $t_c(r)$ then from (20) we have

$$t_c(r) - t_i = \frac{2}{n-1} \frac{r^{\frac{n-1}{2}}}{\sqrt{F(r)}} \quad (21)$$

Now the Kretschmann scalar

$$K = [(n-2)(n-3)+1] \frac{F'^2}{R^{2n-4}R'^2} - 2(n-2)^2(n-3) \frac{FF'}{R^{2n-3}R'^2} + (n-1)(n-2)^2(n-3) \frac{F^2}{R^{2n-2}} \quad (22)$$

diverges at $t = t_c(r)$ i.e., $R = 0$. Thus it represents the formation of a curvature singularity at r . In fact the central singularity (i.e., $r = 0$) forms at the time

$$t_0 = t_i + \sqrt{\frac{2(n-2)}{(n-1)\rho_0}} \quad (23)$$

The Kretschmann scalar also diverges at this central singularity.

Now if we use the expansion (13) for $F(r)$ in equation (21) then near $r = 0$, the singularity curve can be approximately written as (using (23))

$$t_c(r) = t_0 - \frac{F_m}{(n-1)F_0^{3/2}} r^m \quad (24)$$

where $m \geq 2$ and F_m is the first non-vanishing term beyond F_0 . Thus $t_c(r) > t_0$ as $F_m < 0$ for any $m \geq 2$.

To examine whether the singularity at $t = t_0, r = 0$ is naked or not, we investigate whether there exist one or more outgoing null geodesics which terminate in the past at the central singularity. In particular, we will concentrate to radial null geodesics only.

Let us start with the assumption that it is possible to have one or more such geodesics and we choose the form of the geodesics (near $r = 0$) as

$$t = t_0 + ar^\alpha, \quad (25)$$

to leading order in t - r plane with $a > 0, \alpha > 0$. Now for t in the geodesic (25) should be less than $t_c(r)$ in (24) for visibility of the naked singularity so on comparison we have

$$\alpha \geq m \quad \text{and} \quad a < -\frac{F_m}{(n-1)F_0^{3/2}}. \quad (26)$$

Also from the metric form (1), an outgoing null geodesic must satisfy

$$\frac{dt}{dr} = R' \quad (27)$$

But near $r = 0$, the solution (20) for R simplifies to

$$R = r \left[1 - \frac{n-1}{2} \sqrt{F_0} \left(1 + \frac{F_m}{2F_0} r^m \right) t \right]^{\frac{2}{n-1}} \quad (28)$$

Thus combining (25) and (28) in equation (27) we get

$$a\alpha r^{\alpha-1} = \frac{\left[1 - \frac{n-1}{2} \sqrt{F_0} (t_0 + ar^\alpha) - \frac{(2m+n-1)F_m}{4\sqrt{F_0}} r^m (t_0 + ar^\alpha) \right]}{\left[1 - \frac{n-1}{2} \sqrt{F_0} \left(1 + \frac{F_m}{2F_0} r^m \right) (t_0 + ar^\alpha) \right]^{\frac{n-3}{n-1}}} \quad (29)$$

Now if there exists a self consistent solution of this equation then it is possible to have at least one outgoing radial null geodesic that had started at the singularity i.e., the singularity is naked. In order to simplify the above equation we shall use the restrictions in equation (26) in the following two ways:

(i) $\alpha > m$:

The equation (29) becomes (in leading order)

$$a\alpha r^{\alpha-1} = \left(1 + \frac{2m}{n-1} \right) \left(-\frac{F_m}{2F_0} \right)^{\frac{2}{n-1}} r^{\frac{2m}{n-1}} \quad (30)$$

which implies

$$\alpha = 1 + \frac{2m}{n-1} \quad \text{and} \quad a = \left(-\frac{F_m}{2F_0} \right)^{2/(n-1)} \quad (31)$$

Thus for $\alpha > m$ we have $m < \frac{n-1}{n-3}$ and $n > 3$.

For $n = 4$, m may take values 1 and 2 and we have $\rho_1 = F_1 = 0$ and $\rho_2 < 0, F_2 < 0$. As a result, a is real and positive from equations (26) and (31). Moreover, these restrictions are already assumed in the power series expansion for $\rho_i(r)$ so that the initial density gradient is negative and falls off rapidly near the centre. But for $n > 4$, $m = 1$ is the only possible solution for which no real positive solution of ' a ' is permissible from equations (26) and (31). Hence with the restriction $\alpha > m$, we have a consistent solution of equation (29) only for four dimensional space-time i.e., it is possible to have (at least) null geodesics terminate in the past at the singularity only for four dimension and we can have naked singularity for $n = 4$.

(ii) $\alpha = m$:

In this case equation (29) simplifies to

$$mar^{m-1} = \left[-\frac{F_m}{2F_0} - \frac{n-1}{2} \sqrt{F_0} a \right]^{\frac{3-n}{n-1}} \left[-\frac{(2m+n-1)F_m}{2(n-1)F_0} - \frac{n-1}{2} \sqrt{F_0} a \right] r^{\frac{2m}{n-1}} \quad (32)$$

A comparative study of equal powers of r shows that $m = \frac{n-1}{n-3}$ and a depends on F_m and F_0 . Here for $n = 4, m = 3$ and this situation is already discussed by Singh et al [12]. For $n = 5$, we have $m = 2$ and from (32) we get

$$2b^2(4b + \xi) + (2b + \xi)^2 = 0 \quad (33)$$

where $b = \frac{a}{\sqrt{F_0}}$ and $\xi = \frac{F_2}{F_0^2}$.

We note that for real b , we must have $b < -\frac{\xi}{4}$ i.e., $a < -\frac{F_2}{4F_0^{3/2}}$, which is essentially the restriction in (26). It can be shown that the above cubic equation has at least one positive real root if $\xi \leq -(11 + 5\sqrt{5})$. Thus, if $F_2 \leq -(11 + 5\sqrt{5})F_0^2$ we have at least one real positive solution for a which is consistent with equation (29) (or (32)). Further for $n > 5$, we can not have any integral (positive) solution for m and hence equation (32) is not consistent for $n > 5$. So, it is possible for (at least) radial null geodesics which initiate from the singularity and reach to an external observer without get prevented by any trapped surface for $n \leq 5$. Therefore, naked singularity is possible only for four and five dimensions and for higher dimensions ($n \geq 6$) all singularities are covered by trapped surfaces leading to black hole.

Further, to examine whether it is possible to have an entire family of geodesics those have started at the singularity, let us consider geodesics correct to one order beyond equation (25) i.e., of the form

$$t = t_0 + ar^\alpha + dr^{\alpha+\beta} \quad (34)$$

where as before a, d, α and β are positive constants. Thus equation (29) is modified to

$$\alpha ar^{\alpha-1} + (\alpha+\beta)dr^{\alpha+\beta-1} = \frac{\left[1 - \frac{n-1}{2}\sqrt{F_0}(t_0 + ar^\alpha + dr^{\alpha+\beta}) - \frac{(2m+n-1)F_m}{4\sqrt{F_0}}r^m(t_0 + ar^\alpha + dr^{\alpha+\beta})\right]}{\left[1 - \frac{n-1}{2}\sqrt{F_0}\left(1 + \frac{F_m}{2F_0}r^m\right)(t_0 + ar^\alpha + dr^{\alpha+\beta})\right]^{\frac{n-3}{n-1}}} \quad (35)$$

So for $\alpha > m$ (retaining terms upto second order) we have

$$\alpha ar^{\alpha-1} + (\alpha + \beta)dr^{\alpha+\beta-1} = \left(1 + \frac{2m}{n-1}\right) \left(-\frac{F_m}{2F_0}\right)^{\frac{2}{n-1}} r^{\frac{2m}{n-1}} + Dr^{\alpha - \frac{(n-3)m}{n-1}} \quad (36)$$

As before, we have the values of a and α in equation (31) and

$$\beta = 1 - \frac{(n-3)m}{n-1} \quad \text{and} \quad d = \frac{D}{2 + \frac{(5-n)m}{n-1}}$$

with

$$D = \frac{1}{2}\sqrt{F_0} \left(-\frac{F_m}{2F_0}\right)^{\frac{5-n}{n-1}} \left\{(n-3) \left(1 + \frac{2m}{n-1}\right) - n + 1\right\}. \quad (37)$$

As we have similar conclusion as before for $\alpha > m$ so we now consider the case $\alpha = m$. But if we restrict ourselves to the five dimensional case then $m = \alpha = 2$ and we have the same cubic equation (33) for b . Now β can be evaluated from the equation

$$2 + \beta = \frac{2^{5/2}b}{(-\xi - 4b)^{\frac{3}{2}}} \quad (38)$$

and we must have $\beta > 0$, otherwise the geodesics will not lie in the real space-time. As there is no restriction on d so it is totally arbitrary. This implies that there exists an entire family of outgoing null geodesics terminated in the past at the singularity for four and five dimensions only.

IV. DISCUSSION AND CONCLUDING REMARKS

In this paper, we have studied spherical dust collapse in an arbitrary n dimensional space-time. We have considered both non-marginal and marginally bound cases separately in sections II and III respectively. For non-marginal case we have seen that naked singularity may be possible for all dimensions ($n \geq 4$). However, to get a definite conclusion about naked singularity we should study the geodesic equations as it has been done in section III for marginally bound case. But we can not proceed further due to the presence of the complicated hypergeometric functions. Therefore, no definite conclusion is possible for non-marginal case.

On the other hand, for marginally bound case we have definitely concluded that naked singularity is possible only for $n \leq 5$ by studying the existence of radial null geodesic through the singularity. This result also supports our earlier speculation (see ref.[14]). Here, we should mention that this result depends sensitively on the choice of the initial conditions. In fact, if we do not assume the initial density to have an extremum value at the centre (i.e., $\rho_1 \neq 0$) the naked singularity will be possible in all dimensions.

Finally, we should mention that the naked singularity described above is only a local feature, it is not at all a global aspect i.e., it violates the strong form of CCC. For future work it will be nice to consider the non-marginal case more extensively so that some definite conclusion can be drawn regarding the state of the singularity. Also it will be interesting to study gravitational collapse for perfect fluid model.

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